Some Siegel threefolds with a Calabi-Yau model

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Introduction

In the following we describe some examples of Calabi-Yau manifolds that arise as desingularizations of certain Siegel threefolds. Here by a Calabi-Yau manifold we understand a smooth complex projective variety which admits a holomorphic differential form of degree three without zeros and such that the first Betti number is zero. This differential form is unique up to a constant factor, and we call it the Calabi-Yau form. Our interest in this subject is influenced by work of Gritsenko and many discussions with him. The first Siegel modular variety with a Calabi-Yau model and the essentially only one up to now has been discovered by Barth and Nieto. They showed that the "Nieto quintic" $\{x \in P^5(\mathbb{C}), \ \sigma_1(x) = \sigma_5(x) = 0\}$, where σ_i denote the elementary symmetric polynomials, has a Calabi-Yau model and they derived that the Siegel modular variety $A_{1,3}(2)$ of polarization type $(1,3)$ and a certain level two structure has a Calabi-Yau model. Since the Jacobian of a symplectic substitution is $\det(CZ + D)^{-3}$, the Calabi-Yau three-form produces a modular form of weight three and this must be a cusp form, since it survives on a non-singular model as a holomorphic differential form ([Fr], III.2.6). In the paper [GH] Gritsenko and Hulek gave a direct construction of this modular form and obtained a new proof for the fact that $A_{1,3}(2)$ has a Calabi-Yau model. We also refer to [GHSS] for further investigations. Besides this example and some small extensions of this group with the same three-form no other examples of Siegel threefolds with Calabi-Yau model seem to be known. Gritsenko raised the problem of determing all Siegel threefolds which admit a Calabi-Yau model. As we mentioned already, such a threefold will produce a certain cusp form of weight three for the considered modular group Γ. This cusp form has very restrictive properties. Since the induced differential form should have no zero at least at the regular locus of the quotient $\mathbb{H}_2/\Gamma$, all zeros of the form must be contained in the ramification of $\mathbb{H}_2 \to \mathbb{H}_2/\Gamma$. Gritsenko gave examples of such modular forms: we refer to the paper [GC] which contains some systematic study of them. One example that Gritsenko and Cléry describe, is the form ∇_3, which

is a cusp form of weight three for the Hecke group $\Gamma_{2,0}[2]$ with respect to a certain quadratic character χ. Hence a subgroup of index two is a candidate for producing a Calabi-Yau manifold. We will prove that this is the case. We will show more: for any group between $\Gamma_2[2]$ and $\Gamma_{2,0}[2]$ there exists a subgroup of index two (the kernel of χ) which produces a Calabi-Yau manifold.

The modular form ∇_3 will come up in a completely different manner. It is simply the product T of 6 (of the 10 classical) theta constants with suitable properties. In this form it has already been described in [GS] and these expressions occur also in [GN]. This approach has the advantage that we can easily describe the action of the full modular group, which is necessary, since we need information about this form at all boundary components. Another advantage of this description is that we can use the work of Igusa about the structure of the ring of modular forms with respect to his group $\Gamma_2[4,8]$ and of some groups containing this group. Igusa used the ten theta constants of the first kind. If one is concerned with groups in the region of the principal congruence subgroup of level two, there are advantages in using the theta constants of the second kind. We use the very nice approach given by Runge [Ru1], [Ru2].

We also make use of Igusa's method of desingularization of the Siegel threefold with respect to the principal congruence subgroup of level $q > 2$ (we need $q = 4$).

Using Igusa's results or Runge's approach, it is easy to determine the rings of modular forms for the groups in question, and in this way one can produce equations for the Siegel threefolds. The main example is the subgroup of index two of $\Gamma_2[2]$.

In this introduction we only describe the equations of this Siegel threefold in a purely algebraic way.

Theorem. *Let X be the subvariety of $P^5(\mathbb{C})$ given by the intersection of the quartic*

$$y_5^4 = y_0 y_1 y_2 (y_0 + y_1 + y_2 + y_3 + y_4)$$

and the quadric

$$2y_5^2 = y_0 y_1 + y_0 y_2 + y_1 y_2 - y_3 y_4.$$

This is a normal projective variety of dimension three. There exists a desingularization $\tilde{X} \to X$ which is a Calabi-Yau manifold.

The variety X together with the Calabi-Yau form have some symmetries. They are easier to describe in another coordinate system (see 4.5 for the explicit description). In this coordinate system we also will give an explicit algebraic expression for the Calabi-Yau form. We will see:

Theorem. *There is a subgroup $G \subset PGL(5, \mathbb{C})$, isomorphic to the semidirect product $S_3 \cdot (\mathbb{Z}/2\mathbb{Z})^2$, which leaves X and the form ω invariant. For each subgroup H of G the quotient X/H admits a desingularization which is a Calabi-Yau manifold.*

It is not difficult to determine the singular locus of X:

Proposition. *The singular locus of X is the union of 15 smooth curves. It consists of two G-orbits. One orbit consists of three quadrics, the other of 12 lines. Representatives are given by the ideals*

$$(y_0 + y_4, y_1 + y_4, y_3 - y_4, y_2 y_4 + y_5^2), \quad (y_0, y_2, y_3, y_5).$$

The main problem is to find a good resolution of the singularities of X. It might be possible to do this by hand or with the help of a computer. We will find it by interpreting X as a Siegel modular variety.

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1. The Siegel modular group of genus two

Recall that the real symplectic group

$$\mathrm{Sp}(n, \mathbb{R}) = \left\{ M \in \mathrm{GL}(2n, \mathbb{R}); \quad {}^t M I M = I \right\} \qquad \left(I = \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix} \right)$$

acts on the generalized half plane

$$\mathbb{H}_n := \left\{ Z = X + \mathrm{i}Y; \quad Z = {}^t Z, \ Y > 0 \ (\text{positive definite}) \right\}$$

by

$$MZ = (AZ + B)(CZ + D)^{-1}, \qquad M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Let $\Gamma_n := \mathrm{Sp}(n, \mathbb{Z})$ be the Siegel modular group. The principal congruence subgroup of level l iy

$$\Gamma_n[l] := \mathrm{kernel}(\mathrm{Sp}(n, \mathbb{Z}) \longrightarrow \mathrm{Sp}(n, \mathbb{Z}/l\mathbb{Z}))$$

and Igusa's subgroup is

$$\Gamma_n[l, 2l] := \left\{ M \in \Gamma_n[l]; \quad A^t B/l \text{ and } C^t D/l \text{ have even diagonal} \right\}.$$

For even l, Igusa's subgroup is a normal subgroup of Γ_n. Other important subgroups are

$$\Gamma_{n,0}[l] = \{ M \in \Gamma_n; \quad C \equiv 0 \bmod l \}.$$

Theta characteristics in genus two

By definition, a theta characteristic is an element $m = \binom{a}{b}$ from $(\mathbb{Z}/2\mathbb{Z})^{2n}$. Here $a, b \in (\mathbb{Z}/2\mathbb{Z})^n$. The characteristic is called even if ${}^tab = 0$. The group $\mathrm{Sp}(n, \mathbb{Z}/2\mathbb{Z})$ acts on the set of characteristics by

$$M\{m\} := {}^tM^{-1}m + \begin{pmatrix} (A{}^tB)_0 \\ (C{}^tD)_0 \end{pmatrix}.$$

Here S_0 denotes the column built of the diagonal of a square matrix S. It is well-known that $\mathrm{Sp}(n, \mathbb{Z}/2\mathbb{Z})$ acts transitively on the subsets of even and odd characteristics. Recall that for any characteristic the theta function

$$\vartheta[m] = \sum_{g \in \mathbb{Z}^n} e^{\pi \mathrm{i}(Z[g+a/2] + {}^tb(g+a/2))}$$

can be defined. Here we use the identification of $\mathbb{Z}/2\mathbb{Z}$ with the subset $\{0, 1\} \subset \mathbb{Z}$. It vanishes if and only if m is odd. Recall also that the formula

$$\vartheta[M\{m\}](MZ) = v(M, m)\sqrt{\det(CZ + D)}\vartheta[m](Z)$$

holds for $M \in \Gamma_n$, where $v(M, m)$ is a rather delicate 8th root of unity which depends on the choice of the square root.

We are interested in the case $n = 2$. In this case there are ten even characteristics. We will write the coordinates of $\mathbb{H}_2$ as

$$Z = \begin{pmatrix} z_0 & z_1 \\ z_1 & z_2 \end{pmatrix}.$$

A set $\{m_1, m_2, m_3, m_4\}$ of four pairwise different even characteristics is called *syzygetic* if the sum of any three of them is even. There are 15 syzygetic (unordered) quadruples and the group $\mathrm{Sp}(2, \mathbb{Z})$ acts transitively on them. We call

$$\left\{ \begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{matrix} \right\}$$

the standard syzygetic quadruple. We are also interested in the 15 complementary sextuples of even characteristics $\{n_1, \ldots, n_6\}$. From a detailed study of the multipliers $v(M, m)$ one can deduce:

1.1 Lemma.   *Let $\mathfrak{n} = \{n_1, \ldots, n_6\}$ be a sextuple of even characteristics (in the case $n = 2$) complementary to a syzygetic quadruple. Then the product*

$$T = T_{\mathfrak{n}} := \prod_{\nu=1}^{6} \vartheta[n_\nu](Z)$$

is a cusp form of weight 3 for a group conjugate to $\Gamma_{2,0}[2]$ and with respect to a quadratic character $\chi_{\mathfrak{n}}$ on this group. The kernel of this character contains $\Gamma_2[4]$. In the case of the standard tuple the group is $\Gamma_{2,0}[2]$ and the kernel

$$\Gamma_{2,0}[2]_{\mathfrak{n}} := \{ M \in \Gamma_{2,0}[2], \quad \chi_{\mathfrak{n}}(M) = 1 \}$$

is a subgroup of index two.

We describe the character of T for the standard syzygetic quadruple. For this we introduce

$$\Theta := \vartheta \begin{bmatrix} 00 \\ 01 \end{bmatrix} \vartheta \begin{bmatrix} 00 \\ 00 \end{bmatrix} \vartheta \begin{bmatrix} 00 \\ 10 \end{bmatrix} \vartheta \begin{bmatrix} 00 \\ 11 \end{bmatrix}.$$

Since $T \cdot \Theta = \chi_5$ is a modular form with respect to the full modular group, this form also is a modular form for $\Gamma_{2,0}[2]$. The characters of T and Θ differ by the character of χ_5. Using the well-known isomorphism $\Gamma_2/\Gamma_2[2] \cong S_6$, the character of χ_5 corresponds to the sign character of S_6. Hence it is sufficient to describe the character of Θ.

1.2 Lemma. *The character of Θ for $M \in \Gamma_{2,0}[2]$ is given by*

$$(-1)^{(\alpha+\beta+\gamma)/2},$$

where

$$C\,^t D = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix}.$$

The character of T is the product of this character and the only nontrivial character of the full modular group.

The proof can be taken from [SM] (use Lemma 4).

We have to recall the location of the zeros of the theta functions. In the case $^t m = (1, 1, 1, 1)$ the theta function $\vartheta[m]$ has a zero of first order along the diagonal $z_1 = 0$ and every zero component is equivalent to the diagonal mod $\Gamma_2[1, 2]$. We consider the differential form

$$\omega = \omega_{\mathfrak{n}} = T_{\mathfrak{n}} \, dz_0 \wedge dz_1 \wedge dz_2.$$

It is invariant under the group $\Gamma_{2,0}[2]_{\mathfrak{n}}$.

1.3 Proposition. *let $F \subset \mathbb{H}_2$ be an irreducible component of the zero locus of $\omega_{\mathfrak{n}}$. Then there exists an element $M \in \Gamma_2$ whose fixed point set is F. It has the property $M^2 = E$. This element is uniquely determined up to the sign and is actually contained already in*

$$\Gamma_2[2, 4]_{\mathfrak{n}} = \Gamma_2[2, 4] \cap \Gamma_{2,0}[2]_{\mathfrak{n}}.$$

Proof. Taking a conjugate group we may assume that F is the diagonal $z_1 = 0$. Then in the sextuple the characteristic $^t m = (1, 1, 1, 1)$ must occur. The only non trivial modular substitution which fixes the diagonal is $z_1 \mapsto -z_1$. The theta series for $^t m = (1, 1, 1, 1)$ changes its sign under this substitution. The others are invariant. This shows

$$T_{\mathfrak{n}} \begin{pmatrix} z_0 & -z_1 \\ -z_1 & z_2 \end{pmatrix} = -T_{\mathfrak{n}} \begin{pmatrix} z_0 & z_1 \\ z_1 & z_2 \end{pmatrix}.$$

Since this is the transformation law for a modular form of odd weight, the substitution $z_1 \mapsto -z_1$ is in the kernel of $\chi_{\mathfrak{n}}$. It is also in $\Gamma_2[2,4]$.   □

We have to consider the group

$$\Gamma = \Gamma_{\mathfrak{n}} := \big\{ M \in \Gamma_2[2]; \quad \chi_{\mathfrak{n}}(M) = 1 \big\}.$$

This is a subgroup of index two of $\Gamma_2[2]$. Using 1.2 and some computation one can see:

1.4 Lemma.   *For the standard syzygetic quadruple, the group Γ is defined inside $\Gamma_2[2]$ by the condition that the symmetric matrix $C^{\,t}D = \left(\begin{smallmatrix} \alpha & \beta \\ \beta & \gamma \end{smallmatrix} \right)$ has the property $\alpha + \beta + \gamma \equiv 0 \bmod 4$. Moreover $\Gamma = \Gamma_{\mathfrak{n}} \subset \Gamma_2[2]$ is generated by*

1) *The group $\Gamma_2[4]$,*
2) *The elements of $\Gamma_{2,0}[2]_{\mathfrak{n}}$ that are conjugate inside Γ_2 to the diagonal matrix with diagonal $(1, -1, 1, -1)$.*
3) *All elements of $\Gamma_{2,0}[2]_{\mathfrak{n}}$ that are conjugate inside Γ_2 to a translation matrix $\left(\begin{smallmatrix} E & S \\ 0 & E \end{smallmatrix} \right)$ of $\Gamma_2[2]$.*

2. Igusa's desingularization

We consider the principal congruence subgroup $\Gamma_2[l]$ of level $l \geq 3$ and denote by

$$X = X(l) := \overline{\mathbb{H}_2/\Gamma_2[l]}$$

the Satake compactification and by $\tilde{X} = \tilde{X}(l)$ the monoidal transform along the Satake boundary. Igusa proved that $\tilde{X}$ is smooth. The theory of Igusa is very difficult but fortunately we can formulate in a very simple way what we need from it.

Igusa used so-called normal coordinates

$$q_0 = e^{2\pi i(z_0+z_1)/l}, \quad q_2 = e^{2\pi i(z_2+z_1)/l}, \quad q_1 = e^{-2\pi i z_1/l}.$$

We consider them as a holomorphic map

$$\mathbb{H}_2 \longrightarrow \mathbb{C}^3, \quad \begin{pmatrix} z_0 & z_1 \\ z_1 & z_2 \end{pmatrix} \longmapsto (q_0, q_1, q_2).$$

The image is an open subset $D \subset \mathbb{C}^3$. If we denote by $\mathcal{T} = \mathcal{T}(l)$ the group of translations $Z \mapsto Z + lS$, S integral, we get a biholomorphic map

$$\mathbb{H}_2/\mathcal{T} \xrightarrow{\sim} D.$$

We can consider its inverse map and compose it with the projection onto $\mathbb{H}_2/\Gamma_2[l]$ and the inclusion into X to get a holomorphic map

$$D \longrightarrow X.$$

A domain $U \subset \mathbb{C}^n$ is called a *Reinhardt domain* if

$$(z_1, \ldots, z_n) \in U \implies (\zeta_1 z_1, \ldots, \zeta_n z_n) \in U \quad \text{for } |\zeta_\nu| = 1.$$

It is called a *complete Reinhardt domain,* if this is true for all ζ with $|\zeta_\nu| \le 1$. Each Reinhardt domain can be completed to a complete Reinhardt domain in an obvious way. Any holomorphic function on a Reinhardt domain can be expanded in a Laurent series in the whole domain. This Laurent series is a power series if and only if the function extends as a holomorphic function to the completed Reinhardt domain.

2.1 Lemma. *The domain D is a Reinhardt domain $\mathbb{C}^3$. Its completion is*

$$\tilde{D} := D \cup \{q \in \mathbb{C}^3; \ q_0 q_1 q_2 = 0\}.$$

The domain D is dense in $\tilde{D}$.

All we need from Igusa's theory is:

2.2 Theorem (Igusa). *The natural map $D \to X$ extends to a locally biholomorphic map*

$$\tilde{D} \longrightarrow \tilde{X}.$$

The group $\Gamma_2/\Gamma_2[l]$ acts on X and hence on $\tilde{X}$. The translates of the images of $\tilde{D}$ cover $\tilde{X}$.

Each holomorphic function on D can be written as Laurent series in the variables q_ν. Assume that it is the Fourier expansion of a modular form

$$\sum_T a(T) e^{2\pi i \sigma(t_0 z_0 + 2t_1 z_1 + t_2 z_2)/l}, \quad T = \begin{pmatrix} t_0 & t_1 \\ t_1 & t_2 \end{pmatrix}.$$

Here T runs over all matrices such that $t_0, t_2, 2t_1$ are integral. If we rewrite this in the normal coordinates, we get

$$\sum_T a(T) q_0^{t_0} q_1^{t_0 + t_2 - 2t_1} q_2^{t_2}.$$

By Koecher's principle we can have $a(T) \ne 0$ only for semi-positive T. This means $t_0, t_2 \ge 0$ and $t_0 t_2 \ge t_1^2$. This implies $t_0 + t_2 - 2t_1 \ge 0$. Hence the Laurent series actually is a power series. Hence this function extends to a holomorphic function on the whole $\tilde{D}$.

We can talk about the multiplicity of the zero along each of the $q_\nu = 0$. The theta functions $\vartheta[m]$ are periodic under $Z \mapsto 8S$, S integral. Hence we can use $l = 8$ for the normal coordinates. The multiplicities are easy to compute:

2.3 Lemma. *The multiplicity of $\vartheta[m]$ along $q_\nu = 0$ in the coordinates*

$$q_0 = e^{2\pi i(z_0+z_1)/8}, \quad q_2 = e^{2\pi i(z_2+z_1)/8}, \quad q_1 = e^{-2\pi i z_1/8}.$$

is

$$a_1, \quad a_2, \quad a_1 + a_2 - 2a_1 a_2, \qquad m = \begin{pmatrix} a \\ b \end{pmatrix}.$$

Let $m_1, \ldots, m_4$ be a syzygetic quadruple of even characteristics and let $\mathfrak{n} = \{n_1, \ldots, n_6\}$ be the complementary even characteristics. We introduced the modular form

$$T = T_{\mathfrak{n}} = \vartheta[n_1] \cdots \vartheta[n_6].$$

By 1.1 this is a cusp form of weight three for a conjugate of the group $\Gamma_0[2]$. Its character is trivial on $\Gamma_2[4]$. Hence it is better now to use $l = 4$, i.e. to change the notation and to use the normal coordinates

$$q_0 = e^{2\pi i(z_0+z_1)/4}, \quad q_2 = e^{2\pi i(z_2+z_1)/4}, \quad q_1 = e^{-2\pi i z_1/4}.$$

We can consider the differential form

$$\omega = T dz_0 \wedge dz_1 \wedge dz_2$$

on $\tilde{X}(4)$. For general reasons this gives a regular differential form on the whole $\tilde{X}(4)$. We are interested in the zero divisor of this form. We pull it back to $\tilde{D}$ to obtain

$$C \frac{T}{q_0 q_1 q_2} dq_0 dq_1 dq_2.$$

At the moment we only are interested in the behavior of the differential forms along the three divisors $q_\nu = 0$. From 2.3 we can read off the vanishing order of T along $q_\nu = 0$. Of course the result depends on the choice of the syzygetic quadruple. For example the vanishing order of the power series along $q_0 = 0$ is given by

$$\frac{a_{11} + \cdots + a_{61}}{2}, \qquad n_\nu = \begin{pmatrix} a_\nu \\ b_\nu \end{pmatrix}.$$

(The denominator 2 occurs since we used different q_ν in 2.3.) By inspection of the 15 cases one sees that this expression is always 1 or 2. Since for the order of differential form one has to subtract 1, one gets for the vanishing order of the differential form along q_0 either 0 or 1. The same argument works for the variables q_1 and q_2. Now one has to use a list of all the 15 syzygetic quadruples. By inspection one finds:

2.4 Proposition.   *There are* 15 *syzygetic quadruples. We denote the vanishing order of the differential form* ω *(pulled back to* $\tilde{D}$*) at* $q_\nu = 0$ *by* k_ν*. Then one has for* (k_0, k_1, k_2) *the possibilities*

$(0,0,0)$ *(eight cases),*
$(1,1,1)$ *(one case),*
$(0,0,1)$ *(two cases),*
$(0,1,0)$ *(two cases),*
$(1,0,0)$ *(two cases).*

Let us assume that the order along $q_0 = 0$ is one. Then we are in the case $a_{11} + \cdots + a_{61} = 4$. A glance at the power series shows that then only even t_0 occur. Hence the series is invariant under the transformation $q_0 \mapsto -q_0$. This transformation is induced by the translation $z_0 \mapsto z_0 + 2$. Hence this translation belongs to our group $\Gamma_{\mathfrak{n}}$. This translation acts on $\tilde{X}(4)$ as a reflection, meaning that it is of order 2 and fixes an irreducible subvariety of codimension 1. So we obtain the following result which extends 1.3 to the boundary:

2.5 Proposition.   *The modular form T induces on $\tilde{X}(4)$ a differential form with the following property: if $F \subset \tilde{X}$ is an irreducible component of its zero divisor then there exists a reflection in $\Gamma / \pm \Gamma_2[4]$, with fixed point set F. The multiplicities of the zeros are one.*

We need the following general result:

2.6 Theorem.   *Let X be a quasi-projective smooth variety of dimension three and G a finite group of automorphisms of it. Assume that every point of M/G admits an open neighborhood such that on its regular locus exists a three-form without zeros. Then X/G admits a quasi-projective crepant*) desingularization.*

Proof. This theorem is a consequence of theorem 1.2 in [BKS]. Unfortunately the assumptions are slightly different from ours. This theorem only directly applies if an arbitrary stabilizer G_a acts on the tangent space as a subgroup of SL().

For this reason we want to give some details. A cluster is a zero dimensional closed subscheme of M. The set of all clusters of a fixed degree is parametrized by a quasi-projective Hilbert scheme. The set of all G-invariant clusters of degree $\#G$ is a closed subscheme. We denote by G-Hilb(M) the irreducible component which contains the generic clusters (given by $\#G$ distinct points.) Then

$$G\text{-Hilb}(M) \longrightarrow M$$

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*) "Crepant" means that the inverse images of these three-forms have no poles or zeros at the exceptional divisors.

is a crepant resolution. This can be proved analytically locally. Therefore we can assume that $M = \mathbb{C}^3$ and $G \subset \mathrm{GL}(n, \mathbb{C})$ is a linear group. Consider the subgroup $G' \in G$ generated by all pseudo-reflections. By a result of Chevalley $\mathbb{C}^3/G'$ can be identified with $\mathbb{C}^3$ in such a way that $H = G/G'$ acts linearly. Now our assumption gives that H acts by a subgroup of $\mathrm{SL}()$. Hence we can apply the above mentioned theorem 1.2 in [BKR] and obtain that

$$H\text{-}\mathrm{Hilb}(\mathbb{C}^3/G') \longrightarrow \mathbb{C}^3/G$$

is a crepant resolution. Taking inverse images one obtains a natural closed embedding

$$H\text{-}\mathrm{Hilb}(\mathbb{C}^3/G') \longrightarrow G\text{-}\mathrm{Hilb}(\mathbb{C}^3/G).$$

Since it is birational and since G-$\mathrm{Hilb}(\mathbb{C}^3/G)$ is irreducible, it must be an isomorphism.   □

In our main example, concerning the subgroup of index two of $\Gamma_2[2]$, we can avoid using this rather deep theory, since the occurring singularities are rather mild and easy to desingularize. We will describe this below.

Our main result is:

2.7 Theorem.   *Let be Γ' be any group between $\Gamma_{\mathfrak{n}}$ and $\Gamma_0[2]_{\mathfrak{n}}$. Then the Satake compactification of $\mathbb{H}_2/\Gamma'$ admits a desingularization which is a Calabi-Yau manifold.*

It remains to notice that the first Betti number vanishes. It is known that $h^{1,0}$ is zero for all non-singular models of Siegel modular varieties of genus > 1. From Serre duality, it follows that also $h^{2,0} = 0$. This fits to the computation in [LW] . For $\Gamma_2[4]$ one has $h^{2,0} = 6$. It is easy to derive from the description given in [LW] that any form invariant under Γ', is zero.   □

We will give now an explicit and very simple construction of the desingularization in the case of the smallest group $\Gamma = \Gamma_2[2]_{\mathfrak{n}}$. We have to study the singularities of $\tilde{X}(4)/\Gamma$. For this we introduce the finite groups

$$A = \Gamma_2[2]/ \pm \Gamma_2[4] \quad \text{and} \quad B = \Gamma/ \pm \Gamma_2[4].$$

So B is a subgroup of index two of A. The basic point now is that A is an abelian group and that each element of A has order ≤ 2. Igusa [Ig2] proved that $\tilde{X}(4)/A$ is smooth. Let $a \in \tilde{X}(4)$ be some point. Using suitable coordinates the stabilizer A_a can be be linearized. Hence A_a can be considered as some subgroup of $\mathrm{GL}(3, \mathbb{C})$. Since A_a is abelian we can diagonalize it. Hence A_a can be considered as subgroup of $(\mathbb{Z}/2\mathbb{Z})^3$, where this group acts on $\mathbb{C}^3$ in the obvious way by changing signs. Since the quotient by A_a is smooth, A_a must be generated by reflections (here simply sign changes of one variable). Since B is a subgroup of index two of A, we know that B_a is a subgroup of index ≤ 2 of A_a. Hence we obtain:

2.8 Lemma.   *Let $\Gamma = \Gamma_2[2]_\mathfrak{n}$. The quotient $\tilde{X}(4)/\Gamma$ looks locally like $\mathbb{C}^3/H$, where $H \subset (\mathbb{Z}/2\mathbb{Z})^3$ is a subgroup which is contained in some reflection subgroup as subgroup of index ≤ 2.*

The reflection subgroups of $(\mathbb{Z}/2\mathbb{Z})^3$ are trivial to describe. Up to permutation of the variables one has four cases: the trivial subgroup, sign change of the first variable, arbitrary sign changes of the first two variables and sign changes of all variables.

   The group H is a subgroup of index ≤ 2 of such a group. We are only interested in cases where H is not generated by reflections. There are only three types of such groups:

1) The group of order two which is generated by

$$(z_1, z_2, z_3) \longmapsto (-z_1, -z_2, z_3).$$

2) The group of order 4 which is generated by

$$(z_1, z_2, z_3) \longmapsto (-z_1, -z_2, z_3) \quad \text{and} \quad (z_1, z_2, z_3) \longmapsto (z_1, z_2, -z_3).$$

3) The group of order 4 given by sign changes of an even number of coordinates.

In the second case the group H contains a reflection, changing the sign of the third variable. We can take the quotient by this reflection and thus reduce the second case to the first one. So there only two types of singularities occur, case 1) and 3). In both cases the differential form can be written as $h(z)dz_1 \wedge dz_2 \wedge dz_3$ with h some H-invariant holomorphic function in a neighborhood of the origin. Since this differential form should produce a differential form without zeros on the regular locus of the quotient, we obtain that $h(0) \neq 0$. This means that we can assume that ω simply is given by $dz_1 \wedge dz_2 \wedge dz_3$.

   Now we describe the desingularization. In case 1) we blow up $\mathbb{C}^3$ along the line $z_1 = z_2 = 0$. A typical affine chart of the blow-up is given by $(w_1 = z_1/z_2, z_2, z_3)$. Because of

$$z_2 dz_1 \wedge dz_2 \wedge dz_3 = dw_1 \wedge dz_2 \wedge dz_3$$

the pull-back of the differential form $dz_1 \wedge dz_2 \wedge dz_3$ has a zero along $z_2 = 0$. The group H acts now as the group generated by $(w_1, z_2, z_3) \mapsto (w_1, -z_2, z_3)$. This is a reflection group. Hence the quotient is smooth and the zero of ω along the ramification $z_2 = 0$ disappears on the quotient, which gives the desired desingularization of $\mathbb{C}^3/H$.

   The remaining case 3) is slightly more involved. The group H now has order 4 and consists of arbitrary sign changes of an even number of variables. The singular locus is the image of the union of the three coordinate axes in $\mathbb{C}^3$. This time there is no canonical way to desingularize! We have to make a choice. We choose one of the three coordinate axes, for example $z_1 = z_2 = 0$. We start

by blowing up this line. A typical affine chart of the blow up again is $(w_1 = z_1/z_2, z_2, z_3)$. The differential form gets a zero along $z_2 = 0$. Now we have to consider the strict transform of the singular locus. It is given by $w_1 = z_3 = 0$ We have to blow up this locus up again. A typical chart is $(u_1 = w_1/z_3, z_2, z_3)$. The differential form $dz_1 \wedge dz_2 \wedge dz_3$ now gets besides $z_2 = 0$ the additional zero $z_3 = 0$. The group acts on the coordinates $(u_1 = z_1/(z_2 z_3), z_2, z_3)$ by arbitrary sign changes of the variables z_2, z_3. This is a reflection group. The quotient is smooth and the zeros of the differential form disappear on the quotient.

Since in the third case we have no canonical resolution (different choices of the coordinate axes lead to resolutions which are related by flops), we have to explain how to glue the resolutions to get a resolution of the global $\tilde{X}(4)/\Gamma$. The point is that the singular locus of $\tilde{X}(4)/\Gamma$ contains itself only finitely many singular points. These points lead to the case 3). For each of these finitely many points one has to make a choice. But the smooth points of the singular locus lead to case 1) where we have a canonical resolution. Therefore everything fits together to a complex manifold. (This argument does not give projectivity.)

$\square$

3. Equations

In this section we treat the case where $\mathfrak{n}$ is complementary to the standard syzygetic quadruple and Γ means $\Gamma_\mathfrak{n}$ for this choice.

We give the equations for the variety X in the Igusa coordinates. To simplify the notation we will write the theta constants in the form

$$\vartheta[m] = \vartheta\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} \quad \text{for} \quad m = \begin{pmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \end{pmatrix}.$$

The results which we describe now can by taken from Igusa's paper [Ig1] (see page 397).

3.1 Theorem (Igusa). *The five modular forms*

$$y_0 = \vartheta\begin{bmatrix} 00 \\ 11 \end{bmatrix}^4, \quad y_1 = \vartheta\begin{bmatrix} 00 \\ 01 \end{bmatrix}^4, \quad y_2 = \vartheta\begin{bmatrix} 00 \\ 00 \end{bmatrix}^4,$$

$$y_3 = -\vartheta\begin{bmatrix} 10 \\ 00 \end{bmatrix}^4 - \vartheta\begin{bmatrix} 00 \\ 11 \end{bmatrix}^4, \quad y_4 = -\vartheta\begin{bmatrix} 10 \\ 01 \end{bmatrix}^4 - \vartheta\begin{bmatrix} 00 \\ 11 \end{bmatrix}^4$$

generate the ring of modular forms of even weight (with trivial multipliers) for $\Gamma_2[2]$. *The defining relation is the the quartic equation*

$$(y_0 y_1 + y_0 y_2 + y_1 y_2 - y_3 y_4)^2 = 4 y_0 y_1 y_2 (y_0 + y_1 + y_2 + y_3 + y_4).$$

Moreover according to Igusa we have the relation

$$2\vartheta\begin{bmatrix}00\\01\end{bmatrix}^2 \vartheta\begin{bmatrix}00\\00\end{bmatrix}^2 \vartheta\begin{bmatrix}00\\10\end{bmatrix}^2 \vartheta\begin{bmatrix}00\\11\end{bmatrix}^2 = y_0 y_1 + y_0 y_2 + y_1 y_2 - y_3 y_4.$$

We set

$$y_5 := \Theta \qquad \left(= \vartheta\begin{bmatrix}00\\01\end{bmatrix}\vartheta\begin{bmatrix}00\\00\end{bmatrix}\vartheta\begin{bmatrix}00\\10\end{bmatrix}\vartheta\begin{bmatrix}00\\11\end{bmatrix}\right).$$

We recall that the Calabi-Yau form T is the product of the complementary six thetas. The product $y_5 T$ is Igusa's modular form χ_5 which is the unique modular form of weight 5 for the full modular group. The character of this form is non-trivial but trivial on $\Gamma_2[2]$. Hence y_5 is a modular form of weight two with trivial character on our group Γ.

3.2 Proposition.   *The ring of modular forms of even weight for Γ is generated by the six forms of weight two, $y_0, \ldots, y_5$. The defining relations are the quartic*

$$y_5^4 = y_0 y_1 y_2 (y_0 + y_1 + y_2 + y_3 + y_4)$$

and the quadric

$$2y_5^2 = y_0 y_1 + y_0 y_2 + y_1 y_2 - y_3 y_4.$$

Proof.   The field of modular functions for Γ is a quadratic extension of the field of modular functions for $\Gamma_2[2]$. Hence the homogenous field of fractions of $\mathbb{C}[y_0, \ldots, y_5]$ is the full field of modular functions of Γ. Since $y_0, \ldots, y_5$ have no common zero on the Satake compactification, the ring of all modular forms of even weight is the normalization of $\mathbb{C}[y_1, \ldots, y_6]$. Hence it suffices to show that the ideal given by the above two relations is a prime ideal and that the quotient is a normal ring. Since we have two relations, the factor ring is a complete intersection and hence a Cohen-Macaulay ring. Since the singular locus (as has been described in the introduction) is of codimension ≥ 2, on can apply the well-known Serre criterion for normality.   □

4. The Calabi-Yau form

Now we use Runge's approach and consider the theta series of second kind

$$f_a(Z) := \vartheta\begin{bmatrix}a\\0\end{bmatrix}(2Z).$$

They are linked to the $\vartheta[m]$ by the classical relation

$$\vartheta\begin{bmatrix}a\\b\end{bmatrix}(Z)^2 = \sum_{x \in \{0,1\}^2} (-1)^{b'x} f_{a+x}(Z) f_x(Z).$$

Hence the rings $\mathbb{C}[f_a f_b]$ and $\mathbb{C}[\vartheta[m]^2]$ agree. We denote the f_a by f_1, f_2, f_3, f_4 in the ordering

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The following modified version of Igusa's result 3.1 is due to Runge [Ru]:

4.1 Proposition. *The algebra of modular forms of even weight (with trivial multipliers) with respect to the group $\Gamma_2[2]$ is generated by the five forms, all of weight two,*

$$\begin{aligned}
F_1 &= f_1^4 + f_2^4 + f_3^4 + f_4^4, \\
F_2 &= f_1^2 f_2^2 + f_3^2 f_4^2, \\
F_3 &= f_1^2 f_3^2 + f_2^2 f_4^2, \\
F_4 &= f_1^2 f_4^2 + f_2^2 f_3^2, \\
F_5 &= f_1 f_2 f_3 f_4.
\end{aligned}$$

The defining relation is

$$16 F_5^4 = -F_1^2 F_5^2 + F_1 F_2 F_3 F_4 - F_2^2 F_3^2 - F_2^2 F_4^2 + 4 F_2^2 F_5^2 - F_3^2 F_4^2 + 4 F_3^2 F_5^2 + 4 F_4^2 F_5^2.$$

It is very easy to describe the action of $\Gamma_{2,0}[2]$ on the generators:

4.2 Lemma. *The three translations*

$$Z \longmapsto Z + S; \quad S = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

act by $(F_1, \ldots, F_5) \longmapsto$

$$(F_1, -F_2, F_3, -F_4, F_5), \quad (F_1, F_2, F_3, F_4, -F_5), \quad (F_1, F_2, -F_3, -F_4, F_5).$$

The unimodular substitutions

$$Z \longmapsto Z[U]; \quad U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

act by

$$(F_1, \ldots, F_5) \longmapsto (F_1, F_3, F_2, F_4, F_5), \quad (F_1, F_2, F_4, F_3, F_5).$$

The analogue of 3.2 is:

4.3 Proposition. *The ring of modular forms of even weight (with trivial multipliers) of Γ is generated by six modular forms all of weight two, namely $F_1, \ldots, F_5$ (see 4.1) and the additional form $F_6 = \Theta$. There are two defining relations, namely the relation described in 4.1 and the additional quadratic relation*

$$F_6^2 = F_1^2 - 4F_2^2 - 4F_3^2 - 4F_4^2 + 32F_5^2.$$

We know the action of $\Gamma_{2,0}[2]$ also on F_6 (1.2). We give just one example:

4.4 Lemma. *The matrix*

$$M = \begin{pmatrix} E & 0 \\ C & E \end{pmatrix} \quad C = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$$

acts by

$$(F_1, \ldots, F_6) \longmapsto (F_1, F_2, F_3, F_4, F_5, -F_6).$$

In the new coordinates we compute an algebraic expression for the Calabi-Yau differential form

$$\omega := T dz_0 \wedge dz_1 \wedge dz_2, \qquad Z = \begin{pmatrix} z_0 & z_1 \\ z_1 & z_2 \end{pmatrix}.$$

As we mentioned $Ty_5 = \chi_5$ is the well-known cusp form of weight 5 for the full modular group. We will use the homogenous Jacobian

$$W(f_1, f_2, f_3, f_4) = \det \begin{pmatrix} f_1 & f_2 & f_3 & f_4 \\ \partial_0 f_1 & \partial_0 f_2 & \partial_0 f_3 & \partial_0 f_4 \\ \partial_1 f_1 & \partial_1 f_2 & \partial_1 f_3 & \partial_1 f_4 \\ \partial_2 f_1 & \partial_2 f_2 & \partial_2 f_3 & \partial_2 f_4 \end{pmatrix}.$$

Here ∂_i denotes differentiation by z_i. The connection with the usual Jacobian

$$J(f_1/f_4, f_2/f_4, f_3/f_4) = \det \begin{pmatrix} \partial_0(f_1/f_4) & \partial_0(f_2/f_4) & \partial_0(f_3/f_4) \\ \partial_1(f_1/f_4) & \partial_1(f_2/f_4) & \partial_1(f_3/f_4) \\ \partial_2(f_1/f_4) & \partial_2(f_2/f_4) & \partial_2(f_3/f_4) \end{pmatrix}$$

is

$$W(f_1, f_2, f_3, f_4) = f_4^4 J(f_1/f_4, f_2/f_4, f_3/f_4).$$

The Jacobian of a modular substitution M is $\det(CZ + D)^{-3}$. Hence J is a modular form of weight 3 and W is a modular form of weight 5. If one applies a modular transformation to the f_i one obtains a linear transformation of them. This shows that W is invariant under M up to the determinant of this linear transformation. This shows that W up to a constant factor equals

Igusa's modular form χ_5 which is the only modular form of weight 5 for the full modular group. It can be defined as the product of the ten theta series. As a consequence we obtain

$$d(f_1/f_4) \wedge d(f_2/f_4) \wedge d(f_3/f_4) = c \, \frac{\chi_5}{f_4^4} \, dz_0 \wedge dz_1 \wedge dz_2$$

with a certain constant c. Using $T\Theta = \chi_5$ we get

$$c\omega = \frac{f_4^4}{\Theta} \, d(f_1/f_4) \wedge d(f_2/f_4) \wedge d(f_3/f_4).$$

We set

$$g_1 = \frac{f_1}{f_4}, \quad g_2 = \frac{f_2}{f_4}, \quad g_3 = \frac{f_3}{f_4}$$

and, with the notations of 4.1

$$G_1 = \frac{F_2}{F_5}, \quad G_2 = \frac{F_3}{F_5}, \quad G_3 = \frac{F_4}{F_5}$$

We get

$$G_1 = \frac{g_1 g_2}{g_3} + \frac{g_3}{g_1 g_2}, \quad G_2 = \frac{g_1 g_3}{g_2} + \frac{g_2}{g_1 g_3}, \quad G_3 = \frac{g_2 g_3}{g_1} + \frac{g_1}{g_2 g_3}.$$

The Jacobian of this rational transformation is

$$4 \frac{(g_3^2 - g_1^2 g_2^2)(g_2^2 - g_1^2 g_3^2)(g_1^2 - g_2^2 g_3^2)}{g_1^4 g_2^4 g_3^4}$$

or

$$4 \frac{(f_1^2 f_2^2 - f_3^2 f_4^2)(f_1^2 f_3^2 - f_2^2 f_4^2)(f_1^2 f_4^2 - f_2^2 f_3^2)}{f_1^4 f_2^4 f_3^4}$$

This gives

$$4c\omega = \frac{f_1^4 f_2^4 f_3^4 f_4^4 \, d(F_2/F_5) \wedge d(F_3/F_5) \wedge d(F_4/F_5)}{\Theta(f_1^2 f_2^2 - f_3^2 f_4^2)(f_1^2 f_3^2 - f_2^2 f_4^2)(f_1^2 f_4^2 - f_2^2 f_3^2)}$$

$$= \frac{F_5^4}{(F_2 F_3 F_4 - 2F_1 F_5^2)F_6} \, d(F_2/F_5) \wedge d(F_3/F_5) \wedge d(F_4/F_5).$$

The group $\Gamma_{\mathfrak{n}}$ is normal in $\Gamma_{2,0}[2]$. The quotient acts on the ring of modular forms for $\Gamma_{\mathfrak{n}}$. The subgroup of index two

$$G = \Gamma_{2,0}[2]_{\mathfrak{n}}/\Gamma_2[2]_{\mathfrak{n}}$$

leaves the differential form invariant. Using 4.4 and 4.2 we can express the action of G on the generators. We obtain:

4.5 Theorem.   *Let X be the subvariety of $P^5(\mathbb{C})$ given a as intersection of the quartic*

$$16x_4^4 + x_0^2 x_4^2 + x_1^2 x_2^2 + x_1^2 x_3^2 + x_2^2 x_3^2 = x_0 x_1 x_2 x_3 + 4x_4^2(x_1^2 + x_2^2 + x_3^2)$$

and the quadric

$$x_5^2 = x_0^2 - 4x_1^2 - 4x_2^2 - 4x_3^2 + 32x_4^2.$$

This is a normal projective variety of dimension three. There exists a desingularization $\tilde{X} \to X$ which is a Calabi-Yau manifold. The differential form without zeros is given by

$$\omega := \frac{x_4^4}{(x_1 x_2 x_3 - 2x_0 x_4^2)x_5} \, d(x_1/x_4) \wedge d(x_2/x_4) \wedge d(x_3/x_4).$$

The group $G \cong S_3 \cdot (\mathbb{Z}/2\mathbb{Z})^3$ (semidirect product) generated by

1) *arbitrary permutations of x_1, x_2, x_3 followed by the sign change of x_5 if the permutation is odd,*
2) *arbitrary sign changes of two of the x_1, x_2, x_3,*
3) *the sign change of x_4,*

is a group of automorphisms of X which fixes ω. For each subgroup $H \subset G$ the quotient X/H admits a Calabi-Yau desingularization with Calabi-Yau form ω.

For sake of completeness we give the coordinate transformation between the coordinates x_i and the coordinates y_i which we used in the introduction. We have $y_5 = x_5$ and

$$\begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} 1 & -2 & -2 & 2 & 0 \\ 1 & -2 & 2 & -2 & 0 \\ 1 & 2 & 2 & 2 & 0 \\ -1 & 2 & -2 & -2 & -8 \\ -1 & 2 & -2 & -2 & 8 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

The expression we obtained for ω and for the action of G in the coordinates y_i looked not very nice, hence we skip them.

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